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Operators whose dual has non-separable range ☆

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Abstract

Let X and Y be separable Banach spaces and $T : X \to Y$ be a bounded linear operator. We characterize the non-separability of $T^*(Y^*)$ by means of fixing properties of the operator T.

Keywords: Operators; Trees; Schauder bases

1. Introduction

The study of fixing properties of certain classes of operators[1] between separable Banach spaces is a heavily investigated part of Banach Space Theory which is closely related to some central questions, most notably with the problem of classifying, up to isomorphism, all complemented subspaces of classical function spaces (see [28] for an excellent exposition).

Typically, one has an operator $T : X \to Y$ which is "large" in a suitable sense and tries to find a concrete object that the operator T preserves. Various versions of this problem have been studied in the literature and several satisfactory answers have been obtained; see, for instance, [1,4,5,13–16,23,24]. Among them, there are two fundamental results that deserve special attention. The first one is due to A. Pełczyński and asserts that every non-weakly compact operator $T : C[0,1] \to Y$ must fix a copy[2] of c_0. The second result is due to H. P. Rosenthal and asserts

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[1] Throughout the paper by the term *operator* we mean bounded, linear operator.

[2] An operator $T : X \to Y$ is said to *fix a copy* of a Banach space E if there exists a subspace Z of X which is isomorphic to E and is such that $T|_Z$ is an isomorphic embedding.

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that every operator $T : C[0,1] \to Y$ whose dual T^* has non-separable range must fix a copy of $C[0,1]$.

The present paper is a continuation of this line of research and is devoted to the study of the following problem.

Problem 1. Let X and Y be separable Banach spaces and $T : X \to Y$ be an operator such that T^* has non-separable range. What kind of fixing properties does the operator T have?

To state our main results we need to fix some pieces of notation and introduce some terminology. By $2^{<\mathbb{N}}$ we shall denote the Cantor tree. By $\varphi : 2^{<\mathbb{N}} \to \mathbb{N}$ we denote the unique bijection satisfying $\varphi(s) < \varphi(t)$ if either $|s| < |t|$ or $|s| = |t| = n$ and $s <_{\mathrm{lex}} t$ (here $<_{\mathrm{lex}}$ stands for the usual lexicographical order on 2^n). We recall the following class of basic sequences (see [2,3,10]).

Definition 1. Let X be a Banach space and $(x_t)_{t\in 2^{<\mathbb{N}}}$ be a sequence in X indexed by the Cantor tree. We say that $(x_t)_{t\in 2^{<\mathbb{N}}}$ is *topologically equivalent to the basis of James tree* if the following are satisfied.

(1) If (t_n) is the enumeration of $2^{<\mathbb{N}}$ according to the bijection φ, then the sequence (x_{t_n}) is a seminormalized basic sequence.
(2) For every infinite antichain A of $2^{<\mathbb{N}}$ the sequence $(x_t)_{t\in A}$ is weakly null.
(3) For every $\sigma \in 2^{\mathbb{N}}$ the sequence $(x_{\sigma|n})$ is weak* convergent to an element $x^{**}_\sigma \in X^{**} \setminus X$. Moreover, if $\sigma, \tau \in 2^{\mathbb{N}}$ with $\sigma \neq \tau$, then $x^{**}_\sigma \neq x^{**}_\tau$.

The archetypical example of such a sequence is the standard unit vector basis of James tree space JT (see [17]). There are also classical Banach spaces having a natural Schauder basis topologically equivalent to the basis of James tree; the space $C[0,1]$ is an example.

We now introduce the following definition.

Definition 2. Let X and Y be Banach spaces and $T : X \to Y$ be an operator. We say that T *fixes a copy of a sequence topologically equivalent to the basis of James tree* if the there exists a sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ in X such that both $(x_t)_{t\in 2^{<\mathbb{N}}}$ and $(T(x_t))_{t\in 2^{<\mathbb{N}}}$ are topologically equivalent to the basis of James tree.

We notice that if $T : X \to Y$ fixes a copy of a sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ as above, then the topological structure of the weak* closure of $\{x_t \colon t \in 2^{<\mathbb{N}}\}$ in X^{**} is preserved under the action of the operator T^{**} (see Lemma 30). We point out, however, that metric properties are not necessarily preserved (see §5.3).

We are ready to state the first main result of the paper.

Theorem 3. *Let X be a separable Banach space not containing a copy of ℓ_1, Y be a separable Banach space and $T : X \to Y$ be an operator. Then the following are equivalent.*

(i) *The dual operator $T^* : Y^* \to X^*$ of T has non-separable range.*
(ii) *The operator T fixes a copy of a sequence topologically equivalent to the basis of James tree.*

The assumption in Theorem 3 that the space X does not contain a copy of ℓ_1 is not redundant. Indeed, if $Q : \ell_1 \to JT$ is a quotient map, then the dual operator Q^* of Q has non-separable range yet Q is strictly singular[3] and fixes no copy of a sequence topologically equivalent to the basis of James tree. Observe, however, that in this case there exists a bounded sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ in ℓ_1 such that its image $(Q(x_t))_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree. On the other hand, if $Q : \ell_1 \to C[0,1]$ is a quotient map, then Q fixes a copy of ℓ_1. Our second main result shows that this phenomenon holds true in full generality.

Theorem 4. *Let X be a separable Banach space containing a copy of ℓ_1, Y be a separable Banach space and $T : X \to Y$ be an operator. Then the following are equivalent.*

(i) *The dual operator $T^* : Y^* \to X^*$ of T has non-separable range.*
(ii) *Either the operator T fixes a copy of ℓ_1 or there exists a bounded sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ in X such that its image $(T(x_t))_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree.*

The paper is organized as follows. In Section 2 we gather some background material. In Section 3 we give the proof of Theorem 3 while in Section 4 we give the proof of Theorem 4. Finally, in Section 5 we make some comments.

2. Background material

Our general notation and terminology is standard as can be found, for instance, in [19] and [20]. For every Banach space X by B_X we denote the closed unit ball of X. By $\mathbb{N} = \{0, 1, 2, \dots\}$ we shall denote the natural numbers. If S is a countable infinite set, then by $[S]^\infty$ we shall denote the set of all infinite subsets of S. Notice that $[S]^\infty$ is a G_δ, hence Polish, subspace of 2^S.

We will frequently need to compute the descriptive set-theoretic complexity of various sets and relations. To this end, we will use the "Kuratowski–Tarski algorithm". We will assume that the reader is familiar with this classical method. For more details we refer to [19, p. 353].

2.1. Subtrees of the Cantor tree

As we have already mentioned, by $2^{<\mathbb{N}}$ we shall denote the Cantor tree; i.e. $2^{<\mathbb{N}}$ is the set of all finite sequences of 0's and 1's (the empty sequence is denoted by $\emptyset$ and is included in $2^{<\mathbb{N}}$). We view $2^{<\mathbb{N}}$ as a tree equipped with the (strict) partial order $\sqsubset$ of extension. The *length* of a node $t \in 2^{<\mathbb{N}}$ is defined to be the cardinality of the set $\{s \in 2^{<\mathbb{N}}\colon s \sqsubset t\}$ and is denoted by $|t|$. Two nodes $s, t \in 2^{<\mathbb{N}}$ are said to be *comparable* if either $s \sqsubseteq t$ or $t \sqsubseteq s$. Otherwise, s and t are said to be *incomparable*. A subset of $2^{<\mathbb{N}}$ consisting of pairwise comparable nodes is said to be a *chain*, while a subset of $2^{<\mathbb{N}}$ consisting of pairwise incomparable nodes is said to be an *antichain*. For every $s, t \in 2^{<\mathbb{N}}$ we let $s \wedge t$ be the $\sqsubset$-maximal node w of $2^{<\mathbb{N}}$ with $w \sqsubseteq s$ and $w \sqsubseteq t$. If $s, t \in 2^{<\mathbb{N}}$ are incomparable with respect to $\sqsubseteq$, then we write $s \prec t$ provided that $(s \wedge t)^\frown 0 \sqsubseteq s$ and $(s \wedge t)^\frown 1 \sqsubseteq t$. For every $\sigma \in 2^{\mathbb{N}}$ and every $n \in \mathbb{N}$ with $n \geqslant 1$ we set $\sigma \mid n = (\sigma(0), \dots, \sigma(n-1))$ while $\sigma \mid 0 = \emptyset$.

[3] Actually, every operator $T : \ell_1 \to JT$ is strictly singular since every infinite-dimensional subspace of JT contains a copy of ℓ_2 (see [17]).

2.1.1. Downwards closed subtrees

A non-empty subset R of $2^{<\mathbb{N}}$ is said to be a *downwards closed subtree* if for every $t \in R$ and every $s \in 2^{<\mathbb{N}}$ with $s \sqsubseteq t$ we have that $s \in R$. The *body* of a downwards closed subtree R of $2^{<\mathbb{N}}$ is defined to be the set $\{\sigma \in 2^{\mathbb{N}}\colon\ \sigma \mid n \in R\ \forall n \in \mathbb{N}\}$ and is denoted by $[R]$. If A is a non-empty subset of $2^{<\mathbb{N}}$, then the *downwards closure* of A is defined to be the set $\{s \in 2^{<\mathbb{N}}\colon\ \exists t \in A$ with $s \sqsubseteq t\}$ and is denoted by $\hat{A}$; notice that $\hat{A}$ is a downwards closed subtree.

2.1.2. Dyadic subtrees

A subset D of $2^{<\mathbb{N}}$ is said to be a *dyadic subtree* if D can be written in the form $\{s_t\colon\ t \in 2^{<\mathbb{N}}\}$ so that for every $t_0, t_1 \in 2^{<\mathbb{N}}$ we have $t_0 \sqsubset t_1$ (respectively $t_0 \prec t_1$) if and only if $s_{t_0} \sqsubset s_{t_1}$ (respectively $s_{t_0} \prec s_{t_1}$). It is easy to see that such a representation of D as $\{s_t\colon\ t \in 2^{<\mathbb{N}}\}$ is unique. In the sequel when we write $D = \{s_t\colon\ t \in 2^{<\mathbb{N}}\}$, where D is a dyadic subtree, we will assume that this is the canonical representation of D described above. The notion of length and the binary relation $\wedge$ can be relativized to any dyadic subtree D. In particular, if $s \in D$, then we let $|s|_D$ be the cardinality of the set $\{s' \in D\colon\ s' \sqsubset s\}$; moreover, for every $s, s' \in D$ we let $s \wedge_D s'$ be the $\sqsubset$-maximal node $w \in D$ such that $w \sqsubseteq s$ and $w \sqsubseteq s'$. Notice that if $\{s_t\colon\ t \in 2^{<\mathbb{N}}\}$ is the canonical representation of D, then for every $t, t' \in 2^{<\mathbb{N}}$ we have $|s_t|_D = |t|$ and $s_t \wedge_D s_{t'} = s_{t \wedge t'}$.

2.1.3. Regular dyadic subtrees

A dyadic subtree $D = \{s_t\colon\ t \in 2^{<\mathbb{N}}\}$ is said to be *regular* if for every $t_0, t_1 \in 2^{<\mathbb{N}}$ we have $|t_0| = |t_1|$ if and only if $|s_{t_0}| = |s_{t_1}|$; equivalently, the dyadic subtree $D = \{s_t\colon\ t \in 2^{<\mathbb{N}}\}$ is regular if for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\{s_t\colon\ t \in 2^n\} \subseteq 2^m$.

2.2. Families of infinite sets and related combinatorics

Throughout this subsection S will be a countable infinite set. A family $\mathcal{A} \subseteq [S]^\infty$ is said to be *hereditary* if for every $A \in \mathcal{A}$ and every $A' \in [A]^\infty$ we have that $A' \in \mathcal{A}$.

Given $A, B \in [S]^\infty$ we write $A \subseteq^* B$ if the set $A \setminus B$ is finite, while we write $A \perp B$ if the set $A \cap B$ is finite. Two families $\mathcal{A}, \mathcal{B} \subseteq [S]^\infty$ are said to be *orthogonal* if $A \perp B$ for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$. A family $\mathcal{A}$ is said to be *countably generated* in a family $\mathcal{B}$ if there exists a sequence (B_n) in $\mathcal{B}$ such that for every $A \in \mathcal{A}$ there exists $n \in \mathbb{N}$ with $A \subseteq^* B_n$. A subfamily $\mathcal{B}$ of a family $\mathcal{A}$ is said to be *cofinal* in $\mathcal{A}$ if for every $A \in \mathcal{A}$ there exists $B \in [A]^\infty$ with $B \in \mathcal{B}$.

For every $\mathcal{A} \subseteq [S]^\infty$ we set

$$\mathcal{A}^\perp = \big\{B \in [S]^\infty\colon\ B \perp A \text{ for every } A \in \mathcal{A}\big\}. \tag{1}$$

The family $\mathcal{A}^\perp$ is called the *orthogonal* of $\mathcal{A}$. Clearly $\mathcal{A}^\perp$ is hereditary. Moreover, it is invariant under finite changes; that is, if $B \in \mathcal{A}^\perp$ and $C \in [S]^\infty$ are such that $B \triangle C$ is finite, then $C \in \mathcal{A}^\perp$.

We recall the following class of hereditary families introduced in [11].

Definition 5. We say that a hereditary family $\mathcal{A}$ of infinite subsets of S is an *M-family* if for every sequence (A_n) in $\mathcal{A}$ there exists $A \in \mathcal{A}$ whose all but finitely many elements are in $\bigcup_{n \geqslant k} A_n$ for every $k \in \mathbb{N}$.

The notion of an M-family is the "hereditary" analogue of the notion of a *happy family* (also known as *selective co-ideal*) introduced by A.R.D. Mathias [21]. We isolate, for future use, the following easy fact (see [11, Fact 3]).

Fact 6. *Let $\mathcal{A} \subseteq [S]^\infty$ be a hereditary family. Then the following are equivalent.*

(i) *The family $\mathcal{A}$ is an M-family.*
(ii) *For every sequence (A_n) in $\mathcal{A}$ there exists $A \in \mathcal{A}$ such that $A \cap A_n \neq \emptyset$ for infinitely many $n \in \mathbb{N}$.*

Much of our interest on M-families stems from the fact that they possess strong structural properties. To state the particular property we need, we recall the following notion.

Definition 7. Let $\mathcal{A}, \mathcal{B} \subseteq [S]^\infty$ be two hereditary and orthogonal families. A *perfect Lusin gap* inside $(\mathcal{A}, \mathcal{B})$ is a continuous, one-to-one map $2^{\mathbb{N}} \ni \sigma \mapsto (A_\sigma, B_\sigma) \in \mathcal{A} \times \mathcal{B}$ such that the following are satisfied.

(1) For every $\sigma \in 2^{\mathbb{N}}$ we have $A_\sigma \cap B_\sigma = \emptyset$.
(2) For every $\sigma, \tau \in 2^{\mathbb{N}}$ with $\sigma \neq \tau$ we have $(A_\sigma \cap B_\tau) \cup (A_\tau \cap B_\sigma) \neq \emptyset$.

The notion of a perfect Lusin gap is due to S. Todorcevic [29] though it can be traced on earlier work of K. Kunen.

It is relatively easy to see that if $\mathcal{A}, \mathcal{B} \subseteq [S]^\infty$ are hereditary and orthogonal families and there exists a perfect Lusin gap inside $(\mathcal{A}, \mathcal{B})$, then $\mathcal{A}$ is *not* countably generated in $\mathcal{B}^\perp$. We will need the following theorem which establishes the converse for certain pairs of orthogonal families (see [11, Theorem I]).

Theorem 8. *Let $\mathcal{A}, \mathcal{B} \subseteq [S]^\infty$ be two hereditary and orthogonal families. Assume that $\mathcal{A}$ is analytic*[4] *and that $\mathcal{B}$ is an M-family and C-measurable*[5]. *Then, either*

(i) *$\mathcal{A}$ is countably generated in $\mathcal{B}^\perp$, or*
(ii) *there exists a perfect Lusin gap inside $(\mathcal{A}, \mathcal{B})$.*

2.3. Increasing and decreasing antichains of a regular dyadic tree

We recall the following classes of antichains of the Cantor tree introduced in [2, §3].

Definition 9. Let D be a regular dyadic subtree of the Cantor tree. An infinite antichain (s_n) of D will be called *increasing* if the following conditions are satisfied.

(1) For every $n, m \in \mathbb{N}$ with $n < m$ we have $|s_n|_D < |s_m|_D$.
(2) For every $n, m, l \in \mathbb{N}$ with $n < m < l$ we have $|s_n|_D \leqslant |s_m \wedge_D s_l|_D$.
(3I) For every $n, m \in \mathbb{N}$ with $n < m$ we have $s_n \prec s_m$.

The set of all increasing antichains of D will be denoted by $\mathrm{Incr}(D)$. Respectively, an infinite antichain (s_n) of D will be called *decreasing* if (1) and (2) above are satisfied and condition (3I) is replaced by the following.

(3D) For every $n, m \in \mathbb{N}$ with $n < m$ we have $s_m \prec s_n$.

The set of all decreasing antichains of D will be denoted by $\mathrm{Decr}(D)$.

[4] A subset A of a Polish space X is said to be *analytic* if there exists a Borel map $f : \mathbb{N}^{\mathbb{N}} \to X$ such that $f(\mathbb{N}^{\mathbb{N}}) = A$. The complement of an analytic set is said to be *co-analytic*.

[5] A subset of a Polish space is said to be *C-measurable* if it belongs to the smallest σ-algebra that contains the open sets and is closed under the Souslin operation. All analytic and co-analytic sets are C-measurable (see [19]).

We will need the following stability properties of the above defined classes of antichains (see [2, Lemma 8]).

Lemma 10. *Let D be a regular dyadic subtree of $2^{<\mathbb{N}}$. Then the following hold.*

(i) *Let (s_n) be an infinite antichain of D and (s_{n_k}) be a subsequence of (s_n). If $(s_n) \in \mathrm{Incr}(D)$, then $(s_{n_k}) \in \mathrm{Incr}(D)$. Respectively, if $(s_n) \in \mathrm{Decr}(D)$, then $(s_{n_k}) \in \mathrm{Decr}(D)$.*
(ii) *For every infinite antichain (s_n) of D there exists a subsequence (s_{n_k}) of (s_n) such that either $(s_{n_k}) \in \mathrm{Incr}(D)$ or $(s_{n_k}) \in \mathrm{Decr}(D)$.*
(iii) *We have $\mathrm{Incr}(D) = \mathrm{Incr}(2^{<\mathbb{N}}) \cap D^{\mathbb{N}}$ and $\mathrm{Decr}(D) = \mathrm{Decr}(2^{\mathbb{N}}) \cap D^{\mathbb{N}}$. In particular, if R is a regular dyadic subtree of $2^{<\mathbb{N}}$ with $R \subseteq D$, then $\mathrm{Incr}(R) = \mathrm{Incr}(D) \cap R^{\mathbb{N}}$ and $\mathrm{Decr}(R) = \mathrm{Decr}(D) \cap R^{\mathbb{N}}$.*

Notice that for every regular dyadic subtree D of $2^{<\mathbb{N}}$ the sets $\mathrm{Incr}(D)$ and $\mathrm{Decr}(D)$ are Polish subspaces of $D^{\mathbb{N}}$. We will also need the following partition result (see [2, Theorem 10]).

Theorem 11. *Let D be a regular dyadic subtree of $2^{<\mathbb{N}}$ and $\mathcal{C}$ be an analytic subset of $D^{\mathbb{N}}$. Then there exists a regular dyadic subtree R of $2^{<\mathbb{N}}$ with $R \subseteq D$ and such that either $\mathrm{Incr}(R) \subseteq \mathcal{C}$ or $\mathrm{Incr}(R) \cap \mathcal{C} = \emptyset$. Respectively, there exists a regular dyadic subtree R' of $2^{<\mathbb{N}}$ with $R' \subseteq D$ and such that either $\mathrm{Decr}(R') \subseteq \mathcal{C}$ or $\mathrm{Decr}(R') \cap \mathcal{C} = \emptyset$.*

We notice that Theorem 11 is essentially a consequence of the work of V. Kanellopoulos [18] on Ramsey families of subtrees of the Cantor tree.

2.4. Selection of subsequences

Let X be a separable Banach space and for every $n \in \mathbb{N}$ let (x_k^n) be a weakly null sequence in X. If the dual X^* of X is separable, then there exists a strictly increasing sequence (k_n) in $\mathbb{N}$ such that the sequence $(x_{k_n}^n)$ is also weakly null. This property fails if X does not contain a copy of ℓ_1 and X^* is non-separable (see [2,3]). Nevertheless, for this case we have the following "weak subsequence selection" principle discovered by H.P. Rosenthal (see [27, Theorem 3.6]).

Theorem 12. *Let X be a separable Banach space and $\mathcal{K}$ be a weak* compact subset of X^{**} consisting only of Baire-1 functions.[6] For every $n \in \mathbb{N}$ let (x_k^n) be a sequence in $\mathcal{K}$ which is weak* convergent to an element x_n^{**}. Assume that the sequence (x_n^{**}) is weak* convergent to an element x^{**}. Then there exists a sequence (n_i, k_i) in $\mathbb{N} \times \mathbb{N}$ with $n_i < k_i < n_{i+1}$ for every $i \in \mathbb{N}$ and such that the sequence $(x_{k_i}^{n_i})$ is weak* convergent to x^{**}.*

3. Proof of Theorem 3

This section is devoted to the proof of Theorem 3. The fact that (ii) implies (i) follows from the following general fact.

[6] An element x^{**} of X^{**} is said to be *Baire-1* if x^{**} is a Baire-1 function on X^* when X^* is equipped with the weak* topology.

Lemma 13. *Let X and Y be Banach spaces and $T: X \to Y$ be an operator. Assume that there exists a bounded sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ in X such that its image $(T(x_t))_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree. Then T^* has non-separable range.*

Proof. It is essentially a consequence of the Baire Category Theorem. Indeed assume, towards a contradiction, that there exists a sequence (y_n^*) in Y^* such that the set $\{T^*(y_n^*)\colon n \in \mathbb{N}\}$ is norm-dense in $T^*(Y^*)$. For every $i, m, k \in \mathbb{N}$ we define

$$F_{i,m,k} = \big\{\sigma \in 2^{\mathbb{N}}\colon\ y_i^*\big(T(x_{\sigma|n})\big) \geqslant 2^{-m} \text{ for every } n \geqslant k\big\}.$$

Clearly the set $F_{i,m,k}$ is closed. Using the fact that for every $\sigma \in 2^{\mathbb{N}}$ the sequence $(T(x_{\sigma|n}))$ is weak* convergent to an element $y_\sigma^{**} \in Y^{**} \setminus Y$ and our assumption that the sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ is bounded, we see that

$$2^{\mathbb{N}} = \bigcup_{i,m,k\in\mathbb{N}} F_{i,m,k}.$$

Therefore, there exist $t_0 \in 2^{<\mathbb{N}}$ and $i_0, m_0, k_0 \in \mathbb{N}$ such that

$$\big\{\sigma \in 2^{\mathbb{N}}\colon\ t_0 \sqsubset \sigma\big\} \subseteq F_{i_0,m_0,k_0}.$$

This implies that $y_{i_0}^*(T(x_s)) \geqslant 2^{-m_0}$ for every $s \in 2^{<\mathbb{N}}$ such that $t_0 \sqsubset s$ and $|s| \geqslant k_0$. But the sequence $(T(x_{s_n}))$ is weakly null for every infinite antichain (s_n) in $2^{<\mathbb{N}}$, and in particular, for every infinite antichain (s_n) satisfying $t_0 \sqsubset s_n$ and $|s_n| \geqslant k_0$ for every $n \in \mathbb{N}$. Having arrived to the desired contradiction the proof is completed. □

It remains to show that (i) implies (ii). We need to find a sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ in X such that both $(x_t)_{t\in 2^{<\mathbb{N}}}$ and $(T(x_t))_{t\in 2^{<\mathbb{N}}}$ are topologically equivalent to the basis of James tree. Our strategy is to transform the problem to a discrete one concerning families of infinite sets. This reduction will enable us to apply the machinery presented in Sections 2.2 and 2.3 and eventually construct the sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$.

To this end we argue as follows. *We fix a dense sequence (d_n) in the closed unit ball B_X of X and we set $r_n = T(d_n)$ for every $n \in \mathbb{N}$.* Notice that the sequence (d_n) is weak* dense in $B_{X^{**}}$. By the Odell–Rosenthal Theorem [22] and our assumption that the space X does not contain a copy of ℓ_1, we see that $B_{X^{**}}$ consists only of Baire-1 functions. Let $\mathcal{H}$ be the weak* closure of the set $\{r_n\colon n \in \mathbb{N}\}$ in Y^{**}. Clearly $\mathcal{H}$ is weak* compact. Moreover, we have the following fact.

Fact 14. *The set $\mathcal{H}$ consists only of Baire-1 functions.*

Proof. By the Main Theorem in [26], it is enough to show that for every $N \in [\mathbb{N}]^\infty$ there exists $L = \{l_0 < l_1 < \cdots\} \in [N]^\infty$ such that the sequence (r_{l_n}) is weak Cauchy. If this is not the case, then, by Rosenthal's Dichotomy [25], there would existed $M = \{m_0 < m_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that the sequence (r_{m_n}) is equivalent to the standard unit vector basis of ℓ_1; but then, the sequence (d_{m_n}) would also be equivalent to the standard unit vector basis of ℓ_1, a contradiction. □

By the previous remarks, if we deal with a weak* compact subset of $B_{X^{**}}$ or $\mathcal{H}$, then we have at our disposable all classical machinery for compact subsets of Baire-1 functions discovered in [6,22,26]. In what follows, we will use these results without giving an explicit reference, unless there is some particular need to do so.

We are going to introduce four families of infinite subsets of $\mathbb{N}$ which are naturally associated to the sequences (d_n) and (r_n). These families will play a decisive rôle in the proof. The first one is defined by

$$\mathcal{D} = \big\{L \in [\mathbb{N}]^\infty\text{: the sequence } (d_n)_{n\in L} \text{ is weak}^* \text{ convergent}\big\} \tag{2}$$

while the second one is defined by

$$\mathcal{R} = \big\{L \in [\mathbb{N}]^\infty\text{: the sequence } (r_n)_{n\in L} \text{ is weak}^* \text{ convergent}\big\}. \tag{3}$$

Before we give the definition of the next two families, we will isolate some basic properties of $\mathcal{D}$ and $\mathcal{R}$.

Fact 15. *The families $\mathcal{D}$ and $\mathcal{R}$ are hereditary, co-analytic and cofinal in* $[\mathbb{N}]^\infty$.

Proof. It is clear that both $\mathcal{D}$ and $\mathcal{R}$ are hereditary. It is also easy to see that they are cofinal in $[\mathbb{N}]^\infty$. To see that $\mathcal{D}$ is co-analytic notice that

$$\begin{aligned} L \in \mathcal{D} \quad &\Leftrightarrow \quad \text{the sequence } (d_n)_{n\in L} \text{ is weak Cauchy} \\ &\Leftrightarrow \quad \forall x^* \in B_{X^*}\ \forall \varepsilon > 0\ \exists k \in \mathbb{N} \text{ such that} \\ &\qquad\ \big|x^*(d_n) - x^*(d_m)\big| < \varepsilon \quad \text{for every } n, m \in L \quad \text{with } n, m \geqslant k. \end{aligned}$$

The same argument shows that $\mathcal{R}$ is co-analytic. The proof is completed. □

By Fact 15, we see that the family $\mathcal{D} \cap \mathcal{R}$ is hereditary, co-analytic and cofinal in $[\mathbb{N}]^\infty$. We will need the following stronger property which is essentially a consequence of the deep effective version of the Bourgain–Fremlin–Talagrand Theorem due to G. Debs [7,8].

Lemma 16. *There exists a hereditary, Borel and cofinal subfamily $\mathcal{F}$ of $\mathcal{D} \cap \mathcal{R}$. In particular, the family $\mathcal{F}$ is hereditary, Borel and cofinal in* $[\mathbb{N}]^\infty$.

Proof. We have already observed that $B_{X^{**}}$ consists only of Baire-1 functions and that the sequence (d_n) is dense in $B_{X^{**}}$. As it was explained in [9, Remark 1(2)], by Debs' Theorem [7] (see also [8]) there exists a hereditary, Borel and cofinal subfamily $\mathcal{F}_0$ of $\mathcal{D}$. With the same reasoning, we see that there exists a hereditary, Borel and cofinal subfamily $\mathcal{F}_1$ of $\mathcal{R}$. We set $\mathcal{F} = \mathcal{F}_0 \cap \mathcal{F}_1$. Clearly the family $\mathcal{F}$ is as desired. The proof is completed. □

We proceed to define the next two families we mentioned before. The third one is defined by

$$\mathcal{D}_0 = \big\{L \in [\mathbb{N}]^\infty\text{: the sequence } (d_n)_{n\in L} \text{ is weakly null}\big\}. \tag{4}$$

Finally, we set

$$\mathcal{R}_0 = \big\{L \in [\mathbb{N}]^\infty\text{: the sequence } (r_n)_{n\in L} \text{ is weakly null}\big\}. \tag{5}$$

We isolate, below, some structural properties of $\mathcal{D}_0$ and $\mathcal{R}_0$.

Lemma 17. *Both $\mathcal{D}_0$ and $\mathcal{R}_0$ are hereditary, co-analytic and M-families. Moreover, we have $\mathcal{D}_0 \subseteq \mathcal{R}_0$.*

Proof. It is clear that $\mathcal{D}_0 \subseteq \mathcal{R}_0$ and that $\mathcal{D}_0$ and $\mathcal{R}_0$ are hereditary. Arguing as in the proof of Fact 15, it is easy to see that they are co-analytic. It remains to check that they are M-families. We will argue only for the family $\mathcal{D}_0$ (the case of $\mathcal{R}_0$ is similarly treated). By Fact 6, it is enough to show that for every sequence (A_n) in $\mathcal{D}_0$ there exists $A \in \mathcal{D}_0$ such that $A \cap A_n \neq \emptyset$ for infinitely many $n \in \mathbb{N}$. So, let (A_n) be one. We may assume that $A_n \cap A_m = \emptyset$ if $n \neq m$. For every $n \in \mathbb{N}$ let $\{a_0^n < a_1^n < \cdots\}$ be the increasing enumeration of the set A_n and set $x_k^n = d_{a_k^n}$ for every $k \in \mathbb{N}$. Since $A_n \in \mathcal{D}_0$ the sequence (x_k^n) is weakly null. By Theorem 12, there exists a sequence (n_i, k_i) in $\mathbb{N} \times \mathbb{N}$ with $n_i < k_i < n_{i+1}$ and such that the sequence $(x_{k_i}^{n_i})$ is also weakly null. We set $A = \{a_{k_i}^{n_i}\colon i \in \mathbb{N}\}$. Then $A \in \mathcal{D}_0$ and $A_{n_i} \cap A \neq \emptyset$ for every $i \in \mathbb{N}$. The proof is completed. □

We are about to introduce one more family of infinite subsets of $\mathbb{N}$. Let $\mathcal{F}$ be the family obtained by Lemma 16. We set

$$\mathcal{A} = \mathcal{F} \setminus \mathcal{R}_0. \tag{6}$$

The following lemma is the main technical step towards the proof of Theorem 3.

Lemma 18. *There exists a perfect Lusin gap inside $(\mathcal{A}, \mathcal{D}_0)$.*

Proof. It is clear that $\mathcal{A}$ and $\mathcal{D}_0$ are hereditary and orthogonal. By Lemma 16 and Lemma 17, we see that $\mathcal{A}$ is analytic and $\mathcal{D}_0$ is co-analytic and M-family. By Theorem 8, the proof will be completed once we show that $\mathcal{A}$ is not countably generated in $\mathcal{D}_0^{\perp}$. To show this we will argue by contradiction.

So, assume that there exists a sequence (M_k) in $\mathcal{D}_0^{\perp}$ such that for every $L \in \mathcal{A}$ there exists $k \in \mathbb{N}$ with $L \subseteq^* M_k$. For every $k \in \mathbb{N}$ let $\mathcal{K}_k$ be the weak* closure of the set $\{d_n\colon n \in M_k\}$ in X^{**}.

Claim 19. *For every $k \in \mathbb{N}$ there exist $F_k \subseteq X^*$ finite and $\varepsilon_k > 0$ such that*

$$\mathcal{K}_k \cap W(0, F_k, \varepsilon_k) = \emptyset$$

where $W(0, F_k, \varepsilon_k) = \{x^{**} \in X^{**}\colon |x^{**}(x^*)| < \varepsilon_k$ for every $x^* \in F_k\}$.

Proof of Claim 19. Fix $k \in \mathbb{N}$. It is enough to show that $0 \notin \mathcal{K}_k$. To see this assume, towards a contradiction, that $0 \in \mathcal{K}_k$. Since $\mathcal{K}_k \subseteq B_{X^{**}}$ there exists $N \in [M_k]^\infty$ such that $N \in \mathcal{D}_0$. This contradicts the fact that $M_k \in \mathcal{D}_0^{\perp}$. The proof of Claim 19 is completed. □

Let Z be the norm closure of the linear span of the set

$$F = \bigcup_k F_k.$$

Clearly Z is a norm-separable subspace of X^*.

Claim 20. *We have $T^*(Y^*) \subseteq Z$.*

Granting Claim 20, the proof of Lemma 18 is completed. Indeed, the inclusion $T^*(Y^*) \subseteq Z$ and the norm-separability of Z yield that T^* has separable range. This contradicts our assumption on the operator T.

It remains to prove Claim 20. Again we will argue by contradiction. So, assume that $T^*(Y^*) \nsubseteq Z$. There exist $y^* \in Y^*$, $x^{**} \in X^{**}$ and $\delta > 0$ such that

(a) $\|T^*(y^*)\| = \|x^{**}\| = 1$,
(b) $Z \subseteq \mathrm{Ker}(x^{**})$, and
(c) $x^{**}(T^*(y^*)) > \delta$.

By (a) above, we may select $L \in [\mathbb{N}]^\infty$ such that the sequence $(d_n)_{n\in L}$ is weak* convergent to x^{**}. By (c), we may assume that $y^*(T(d_n)) = y^*(r_n) > \delta$ for every $n \in L$, and so, $[L]^\infty \cap \mathcal{R}_0 = \emptyset$. By Lemma 16, the family $\mathcal{F}$ is hereditary and cofinal in $[\mathbb{N}]^\infty$. Hence, there exists $A \in [L]^\infty$ such that $[A]^\infty \subseteq \mathcal{A}$. Recall that the sequence (M_k) generates $\mathcal{A}$. Therefore, there exists $k_0 \in \mathbb{N}$ such that $A \subseteq^* M_{k_0}$. We select $N \in [A]^\infty$ with $N \subseteq M_{k_0}$. By Claim 19, the set F_{k_0} is finite and $d_n \notin W(0, F_{k_0}, \varepsilon_{k_0})$ for every $n \in N$. Hence, there exist $x_0^* \in F_{k_0}$ and $M \in [N]^\infty$ such that $|x_0^*(d_n)| \geqslant \varepsilon_{k_0}$ for every $n \in M$. Since the sequence $(d_n)_{n\in L}$ is weak* convergent to x^{**} and $M \in [L]^\infty$ we get that $|x^{**}(x_0^*)| \geqslant \varepsilon_{k_0}$. In particular, $x_0^* \notin \mathrm{Ker}(x^{**})$ and $x_0^* \in F_{k_0} \subseteq F \subseteq Z$. This contradicts property (b) above. Having arrived to the desired contradiction, the proof of Claim 20 is completed, and as we have already indicated, the proof of Lemma 18 is also completed. □

We fix a perfect Lusin gap $2^{\mathbb{N}} \ni \sigma \mapsto (A_\sigma, B_\sigma) \in \mathcal{A} \times \mathcal{D}_0$ whose existence is guaranteed by Lemma 18. We recall the following properties of this assignment.

(P1) The map $2^{\mathbb{N}} \ni \sigma \mapsto (A_\sigma, B_\sigma) \in [\mathbb{N}]^\infty \times [\mathbb{N}]^\infty$ is one-to-one and continuous.
(P2) For every $\sigma \in 2^{\mathbb{N}}$ we have $A_\sigma \cap B_\sigma = \emptyset$.
(P3) For every $\sigma, \tau \in 2^{\mathbb{N}}$ with $\sigma \neq \tau$ we have $(A_\sigma \cap B_\tau) \cup (A_\tau \cap B_\sigma) \neq \emptyset$.

Let $\sigma \in 2^{\mathbb{N}}$ be arbitrary. Since $A_\sigma \in \mathcal{A} \subseteq (\mathcal{D} \cap \mathcal{R}) \setminus \mathcal{R}_0$ and $\mathcal{D}_0 \subseteq \mathcal{R}_0$ we see that there exist two non-zero vectors $x_\sigma^{**} \in X^{**}$ and $y_\sigma^{**} \in Y^{**}$ such that

$$x_\sigma^{**} = \text{weak}^* - \lim_{n\in A_\sigma} d_n \quad \text{and} \quad y_\sigma^{**} = \text{weak}^* - \lim_{n\in A_\sigma} r_n. \tag{7}$$

Notice that

$$y_\sigma^{**} = T^{**}\big(x_\sigma^{**}\big). \tag{8}$$

The following lemma is a consequence of properties (P2) and (P3) and it is a typical application of related combinatorics (see, for instance, [12, Lemma 3.2] and the references therein).

Lemma 21. *For every uncountable subset U of $2^{\mathbb{N}}$ there exists a sequence (σ_n) in U such that the sequences $(x_{\sigma_n}^{**})$ and $(y_{\sigma_n}^{**})$ are both weak* convergent to* 0.

Proof. By (8) and the weak* continuity of the operator T^{**}, it is enough to find a sequence (σ_n) in U such that the sequence $(x_{\sigma_n}^{**})$ is weak* convergent to 0. To this end, it suffices to show that 0 belongs to the weak* closure of the set $\{x_\sigma^{**}\colon \sigma \in U\}$ in X^{**}. Assume, towards a contradiction, that this is not the case. It is then possible to select a weak* open subset $\mathcal{W}$ of X^{**} and a weak* closed subset $\mathcal{F}$ of X^{**} such that $0 \in \mathcal{W} \subseteq \mathcal{F}$ and $x^{**}_\sigma \notin \mathcal{F}$ for every $\sigma \in U$. We set

$$A = \{n \in \mathbb{N}\colon\ d_n \notin \mathcal{F}\} \quad \text{and} \quad B = \{n \in \mathbb{N}\colon\ d_n \in \mathcal{W}\} \tag{9}$$

and we notice $A \cap B = \emptyset$. Let $\sigma \in U$ be arbitrary. By (7) and the fact that $x^{**}_\sigma \notin \mathcal{F}$, we see that $A_\sigma \subseteq^* A$. Moreover, since $B_\sigma \in \mathcal{D}_0$ and $0 \in \mathcal{W}$ we have $B_\sigma \subseteq^* B$. Therefore, it is possible to find $k \in \mathbb{N}$ and an uncountable subset U' of U such that

$$A_\sigma \setminus \{0, \dots, k\} \subseteq A \quad \text{and} \quad B_\sigma \setminus \{0, \dots, k\} \subseteq B \tag{10}$$

for every $\sigma \in U'$. There exist two subsets F and G of $\{0, \dots, k\}$ and an uncountable subset U'' of U' such that

$$A_\sigma \cap \{0, \dots, k\} = F \quad \text{and} \quad B_\sigma \cap \{0, \dots, k\} = G \tag{11}$$

for every $\sigma \in U''$. Notice that $F \cap G = \emptyset$; indeed, by (11) and property (P2), for every $\sigma \in U''$ we have $F \cap G \subseteq A_\sigma \cap B_\sigma = \emptyset$.

Let $\sigma, \tau \in U''$ with $\sigma \neq \tau$. By (10) and (11), we see that

$$(A_\sigma \cap B_\tau) \cup (A_\tau \cap B_\sigma) \subseteq (F \cap G) \cup (A \cap B) = \emptyset.$$

This contradicts property (P3). Having arrived to the desired contradiction, the proof is completed. □

We should point out that properties (P2) and (P3) will not be used in the rest of the proof. However, heavy use will be made of property (P1). We proceed with the following lemma.

Lemma 22. *There exists a perfect subset P of $2^{\mathbb{N}}$ such that $x^{**}_\sigma \neq x^{**}_\tau$ and $y^{**}_\sigma \neq y^{**}_\tau$ for every $\sigma, \tau \in P$ with $\sigma \neq \tau$.*

Proof. For every subset S of $2^{\mathbb{N}}$ by $[S]^2$ we denote the set of all unordered pairs of elements of S. We set

$$\mathcal{X} = \big\{\{\sigma, \tau\} \in \big[2^{\mathbb{N}}\big]^2\colon\ x^{**}_\sigma \neq x^{**}_\tau\big\} \quad \text{and} \quad \mathcal{Y} = \big\{\{\sigma, \tau\} \in \big[2^{\mathbb{N}}\big]^2\colon\ y^{**}_\sigma \neq y^{**}_\tau\big\}.$$

The sets $\mathcal{X}$ and $\mathcal{Y}$ are analytic in $[2^{\mathbb{N}}]^2$, in the sense that the sets

$$\big\{(\sigma, \tau) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}}\colon\ \{\sigma, \tau\} \in \mathcal{X}\big\} \quad \text{and} \quad \big\{(\sigma, \tau) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}}\colon\ \{\sigma, \tau\} \in \mathcal{Y}\big\}$$

are both analytic subsets of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. Indeed, by (7), we have

$$\begin{aligned} \{\sigma, \tau\} \in \mathcal{X} \quad &\Leftrightarrow \quad \exists x^* \in B_{X^*}\ \exists \varepsilon > 0\ \exists k \in \mathbb{N} \quad \text{such that } \big|x^*(d_n) - x^*(d_m)\big| \geqslant \varepsilon \\ &\text{for every } n \in A_\sigma \text{ and every } m \in A_\tau \quad \text{with } n, m \geqslant k. \end{aligned}$$

Since the map $2^{\mathbb{N}} \ni \sigma \mapsto A_\sigma \in [\mathbb{N}]^\infty$ is continuous, the above equivalence yields that the set $\mathcal{X}$ is analytic. With the same reasoning and using the continuity of the map $2^{\mathbb{N}} \ni \sigma \mapsto B_\sigma \in [\mathbb{N}]^\infty$

we see that $\mathcal{Y}$ is also analytic. By a result of F. Galvin (see [19, Theorem 19.7]), there exists a perfect subset P of $2^{\mathbb{N}}$ such that one of the following cases occur.

CASE 1: $[P]^2 \cap \mathcal{X} = \emptyset$. In this case we see that there exists a non-zero vector $x^{**} \in X^{**}$ such that $x^{**}_\sigma = x^{**}$ for every $\sigma \in P$. This is impossible by Lemma 21.

CASE 2: $[P]^2 \cap \mathcal{Y} = \emptyset$. As above, we see that there exists a non-zero vector $y^{**} \in Y^{**}$ such that $y^{**}_\sigma = y^{**}$ for every $\sigma \in P$. This is also impossible.

CASE 3: $[P]^2 \subseteq (\mathcal{X} \cap \mathcal{Y})$. Notice that, in this case, we have $x^{**}_\sigma \neq x^{**}_\tau$ and $y^{**}_\sigma \neq y^{**}_\tau$ for every $\sigma, \tau \in P$ with $\sigma \neq \tau$. Therefore, the perfect set P is as desired. The proof is completed. □

So far we have been working with the perfect Lusin gap inside $(\mathcal{A}, \mathcal{D}_0)$. The next lemma we will unable us to start the process for selecting the sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$.

Lemma 23. *Let P be the perfect subset of $2^{\mathbb{N}}$ obtained by Lemma 22. Then there exist a sequence $(k_t)_{t\in 2^{<\mathbb{N}}}$ in $\mathbb{N}$ and a continuous, one-to-one map $h : 2^{\mathbb{N}} \to P$ with the following properties.*

(i) *For every $t, t' \in 2^{<\mathbb{N}}$ with $t \neq t'$ we have $k_t \neq k_{t'}$.*
(ii) *For every $\tau \in 2^{\mathbb{N}}$ we have $\{k_{\tau|n}\colon n \in \mathbb{N}\} \subseteq A_{h(\tau)}$.*

Proof. Every infinite subset of $\mathbb{N}$ is naturally identified with an element of $2^{\mathbb{N}}$. Therefore, the map $P \ni \sigma \mapsto A_\sigma \in 2^{\mathbb{N}}$ is continuous and one-to-one. Let F be its image and denote by $f : F \to P$ its inverse. Notice that F is closed and that f is a homeomorphism. There exists a downwards closed subtree R of $2^{<\mathbb{N}}$ such that $[R] = F$. Observe that R is a perfect subtree; that is, every $t \in R$ has two incomparable extensions in R. Hence, it is possible to select a dyadic subtree $D = \{s_t\colon t \in 2^{<\mathbb{N}}\}$ of $2^{<\mathbb{N}}$ such that $D \subseteq R$ and with the following properties.

(a) For every $t \in 2^{<\mathbb{N}}$ the node s_t ends with 1; that is, there exists $w_t \in 2^{<\mathbb{N}}$ such that $s_t = w_t^\frown 1$.
(c) For every $t, t' \in 2^{<\mathbb{N}}$ with $t \neq t'$ we have $|s_t| \neq |s_{t'}|$.

We set $k_t = |s_t| - 1$ for every $t \in 2^{<\mathbb{N}}$ and we define $h : 2^{\mathbb{N}} \to P$ by the rule

$$h(\tau) = f\Big(\bigcup_{n\in\mathbb{N}} s_{\tau|n}\Big).$$

The sequence $(k_t)_{t\in 2^{<\mathbb{N}}}$ and the map h are as desired. □

Let $(k_t)_{t\in 2^{<\mathbb{N}}}$ be the sequence in $\mathbb{N}$ obtained by Lemma 23. For every $t \in 2^{<\mathbb{N}}$ we define

$$e_t = d_{k_t}. \tag{12}$$

The desired sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ will be a subsequence of $(e_t)_{t\in 2^{<\mathbb{N}}}$ of the form $(e_{s_t})_{t\in 2^{<\mathbb{N}}}$ where $\{s_t\colon t \in 2^{<\mathbb{N}}\}$ is dyadic subtree of $2^{<\mathbb{N}}$. We isolate, for future use, the following immediate consequence of Lemma 23.

(P4) For every $\tau \in 2^{\mathbb{N}}$ the sequences $(e_{\tau|n})$ and $(T(e_{\tau|n}))$ are weak* convergent to the non-zero vectors $x^{**}_{h(\tau)}$ and $y^{**}_{h(\tau)}$ respectively.

Lemma 24. *There exist a regular dyadic subtree D_0 of $2^{<\mathbb{N}}$ and a constant $\theta > 0$ such that $\|T(e_t)\| \geqslant \theta$ for every $t \in D_0$.*

Proof. We will show that there exist $s_0 \in 2^{<\mathbb{N}}$ and $\theta > 0$ such that for every $t \in 2^{<\mathbb{N}}$ with $s_0 \sqsubseteq t$ we have $\|T(e_t)\| \geqslant \theta$. In such a case, the regular dyadic subtree $D_0 = \{t \in 2^{<\mathbb{N}}\colon s_0 \sqsubseteq t\}$ and the constant θ satisfy the requirements of the lemma.

To find the node s_0 and the constant θ we will argue by contradiction. So, assume that for every $s \in 2^{<\mathbb{N}}$ and every $\theta > 0$ there exists $t \in 2^{<\mathbb{N}}$ with $s \sqsubseteq t$ and such that $\|T(e_t)\| \leqslant \theta$. Hence, it is possible to select a sequence (t_k) in $2^{<\mathbb{N}}$ such that for every $k \in \mathbb{N}$ we have

(a) $t_k \sqsubset t_{k+1}$ and
(b) $\|T(e_{t_k})\| \leqslant 2^{-k}$.

By (a) above, the set $\{t_k\colon k \in \mathbb{N}\}$ is an infinite chain. We set

$$\tau = \bigcup_{k\in\mathbb{N}} t_k \in 2^{\mathbb{N}}.$$

By property (P4), the sequence $(T(e_{\tau|n}))$ is weak* convergent to the non-zero vector $y^{**}_{h(\tau)}$. Hence, so is the sequence $(T(e_{t_k}))$. By (b) above, we see that $y^{**}_{h(\tau)} = 0$, a contradiction. The proof is completed. □

Lemma 25. *There exists a regular dyadic subtree D_1 of $2^{<\mathbb{N}}$ with $D_1 \subseteq D_0$ and such that for every infinite antichain A of D_1 the sequence $(e_t)_{t\in A}$ is weakly null.*

Proof. Consider the subset $\mathcal{C}$ of $D_0^{\mathbb{N}}$ defined by

$$(s_n) \in \mathcal{C} \quad \Leftrightarrow \quad \text{the sequence } (e_{s_n}) \quad \text{is weakly null.}$$

It is easy to check that $\mathcal{C}$ is a co-analytic subset of $D_0^{\mathbb{N}}$. Applying Theorem 11 for the increasing antichains of D_0 and the color $\mathcal{C}$, we find a regular dyadic subtree R of $2^{<\mathbb{N}}$ with $R \subseteq D_0$ and such that either $\mathrm{Incr}(R) \subseteq \mathcal{C}$ or $\mathrm{Incr}(R) \cap \mathcal{C} = \emptyset$. Next, applying Theorem 11 for the decreasing antichains of R and the same color, we find a regular dyadic subtree D_1 of $2^{<\mathbb{N}}$ with $D_1 \subseteq R$ and such that either $\mathrm{Decr}(D_1) \subseteq \mathcal{C}$ or $\mathrm{Decr}(D_1) \cap \mathcal{C} = \emptyset$. We will show that the regular dyadic subtree D_1 is the desired one. Indeed, notice that $D_1 \subseteq D_0$. To show that for every infinite antichain A of D_1 the sequence $(e_t)_{t\in A}$ is weakly null, we will show first the following weaker property.

Claim 26. *Either* $\mathrm{Incr}(D_1) \subseteq \mathcal{C}$ *or* $\mathrm{Decr}(D_1) \subseteq \mathcal{C}$.

Proof of Claim 26. Let $\mathcal{K}$ be the weak* closure of the set $\{e_t\colon t \in D_1\}$ in X^{**}. By property (P4), we have that $x^{**}_{h(\tau)} \in \mathcal{K}$ for every $\tau \in [\hat{D}_1]$. The map h is one-to-one. Therefore, the set $U = \{h(\tau)\colon \tau \in [\hat{D}_1]\}$ is uncountable. By Lemma 21, there exists a sequence (τ_n) in $[\hat{D}_1]$ such that the sequence $(x^{**}_{h(\tau_n)})$ is weak* convergent to 0. Hence, $0 \in \mathcal{K}$. It is then possible to select an infinite subset S of D_1 such that the sequence $(e_t)_{t\in S}$ is weakly null. Applying the classical Ramsey Theorem, we find an infinite subset S' of S which is either a chain or an antichain. Notice that S' has to be an antichain (for if not, there would existed $\tau \in [\hat{D}_1]$ such that $x^{**}_{h(\tau)} = 0$). By part (ii) of Lemma 10, there exists a sequence (s_n) in S' such that either $(s_n) \in \mathrm{Incr}(D_1)$ or

$(s_n) \in \mathrm{Decr}(D_1)$. If $(s_n) \in \mathrm{Incr}(D_1)$, then, by part (iii) of Lemma 10, we see that $\mathrm{Incr}(R) \cap \mathcal{C} \neq \emptyset$ and so $\mathrm{Incr}(D_1) \subseteq \mathrm{Incr}(R) \subseteq \mathcal{C}$. Otherwise, $\mathrm{Decr}(D_1) \cap \mathcal{C} \neq \emptyset$ which yields that $\mathrm{Decr}(D_1) \subseteq \mathcal{C}$. The proof of Claim 26 is completed. $\square$

Next we strengthen the conclusion of Claim 26 as follows.

Claim 27. *We have* $\mathrm{Incr}(D_1) \subseteq \mathcal{C}$ *and* $\mathrm{Decr}(D_1) \subseteq \mathcal{C}$.

Proof of Claim 27. By Claim 26, either $\mathrm{Incr}(D_1) \subseteq \mathcal{C}$ or $\mathrm{Decr}(D_1) \subseteq \mathcal{C}$. As the argument is symmetric, we will assume that $\mathrm{Incr}(D_1) \subseteq \mathcal{C}$. Recursively, for every $n \in \mathbb{N}$ we select an infinite antichain (t_k^n) of D_1 such that the following are satisfied.

(a) For every $n \in \mathbb{N}$ we have $(t_k^n) \in \mathrm{Incr}(D_1)$.
(b) For every $n, n' \in \mathbb{N}$ with $n < n'$ and every $k, l \in \mathbb{N}$ we have $t_k^{n'} \prec t_l^n$.

The recursive selection is fairly standard and the details are left to the reader. By (a) above and our assumption that $\mathrm{Incr}(D_1) \subseteq \mathcal{C}$, we see that for every $n \in \mathbb{N}$ the sequence $(e_{t_k^n})$ is weakly null. By Theorem 12, there exists a sequence (n_i, k_i) in $\mathbb{N} \times \mathbb{N}$ with $n_i < k_i < n_{i+1}$ for every $i \in \mathbb{N}$ and such that the sequence $(e_{t_{k_i}^{n_i}})$ is also weakly null. By (b), we see that

(c) $t_{k_{i'}}^{n_{i'}} \prec t_{k_i}^{n_i}$ for every $i, i' \in \mathbb{N}$ with $i < i'$.

By part (ii) of Lemma 10, there exists a subsequence of $(t_{k_i}^{n_i})$, denoted for simplicity by (s_m), such that either $(s_m) \in \mathrm{Incr}(D_1)$ or $(s_m) \in \mathrm{Decr}(D_1)$. Invoking (c), we get that $(s_m) \in \mathrm{Decr}(D_1)$. Since the sequence (e_{s_m}) is weakly null, we conclude that $\mathrm{Decr}(D_1) \cap \mathcal{C} \neq \emptyset$ and so $\mathrm{Decr}(D_1) \subseteq \mathcal{C}$. The proof of Claim 27 is completed. $\square$

We are now ready to check that the sequence $(e_t)_{t \in A}$ is weakly null for every infinite antichain A of D_1. So let A be one. Let B be an arbitrary infinite subset of A. By part (ii) of Lemma 10, there exists an infinite sequence (s_n) in B such that either $(s_n) \in \mathrm{Incr}(D_1)$ or $(s_n) \in \mathrm{Decr}(D_1)$. By Claim 27, we see that the sequence (e_{s_n}) is weakly null. In other words, every subsequence of $(e_t)_{t \in A}$ has a further weakly null subsequence. This yields that the entire sequence $(e_t)_{t \in A}$ is weakly null. Thus, the proof of Lemma 25 is completed. $\square$

As we have already mentioned in the introduction, by $\varphi : 2^{<\mathbb{N}} \to \mathbb{N}$ we denote the unique bijection satisfying $\varphi(t) < \varphi(t')$ if either $|t| < |t'|$ or $|t| = |t'|$ and $t <_{\mathrm{lex}} t'$.

Lemma 28. *There exists a dyadic subtree* $D_2 = \{s_t\colon t \in 2^{<\mathbb{N}}\}$ *of* $2^{<\mathbb{N}}$ *such that* $D_2 \subseteq D_1$ *and with the following property. If* (t_n) *is the enumeration of* $2^{<\mathbb{N}}$ *according to the bijection* φ*, then the sequences* $(e_{s_{t_n}})$ *and* $(T(e_{s_{t_n}}))$ *are both seminormalized basic sequences.*

Proof. Notice, first, that the sequences $(e_t)_{t \in D_1}$ and $(T(e_t))_{t \in D_1}$ are seminormalized. Indeed, $D_1 \subseteq D_0$ and so, by Lemma 24, for every $t \in D_1$ we have

$$\theta \leqslant \|T(e_t)\| \leqslant \|T\| \quad \text{and} \quad \theta \cdot \|T\|^{-1} \leqslant \|e_t\| \leqslant 1.$$

Let $t \in D_1$ be arbitrary. We select an infinite antichain A of D_1 such that $t \sqsubset s$ for every $s \in A$. By Lemma 25, the sequences $(x_s)_{s\in A}$ and $(T(x_s))_{s\in A}$ are both weakly null. Using this observation and the classical procedure of Mazur for selecting basic sequences (see [20]), the result follows. $\square$

Let $D_2 = \{s_t\colon t \in 2^{<\mathbb{N}}\}$ be the dyadic subtree obtained by Lemma 28. For every $t \in 2^{<\mathbb{N}}$ we define

$$x_t = e_{s_t} \tag{13}$$

We will show that the sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ is the desired one.

(1) Let (t_n) be the enumeration of $2^{<\mathbb{N}}$ according to the bijection φ. By Lemma 28, we have that (x_{t_n}) and $(T(x_{t_n}))$ are seminormalized basic sequences.

(2) Let A be an infinite antichain of $2^{<\mathbb{N}}$. Notice that the set $A' = \{s_t\colon t \in A\}$ is an infinite antichain of D_2. Since $D_2 \subseteq D_1$, by Lemma 25, we see that the sequences $(x_t)_{t\in A}$ and $(T(x_t))_{t\in A}$ are both weakly null.

(3) Let $\sigma \in 2^{\mathbb{N}}$ be arbitrary. Observe that the set $\{s_{\sigma|n}\colon n \in \mathbb{N}\}$ is an infinite chain of $2^{<\mathbb{N}}$. We define

$$\tau_\sigma = \bigcup_{n\in\mathbb{N}} s_{\sigma|n} \in 2^{\mathbb{N}}$$

and we notice that $(x_{\sigma|n})$ is a subsequence of $(e_{\tau_\sigma|n})$. By property (P4), we see that the sequences $(x_{\sigma|n})$ and $(T(x_{\sigma|n}))$ are weak* convergent to the non-zero vectors $x^{**}_{h(\tau_\sigma)}$ and $y^{**}_{h(\tau_\sigma)}$ respectively.

Next we check that $x^{**}_{h(\tau_\sigma)} \in X^{**} \setminus X$. Assume on the contrary that $x^{**}_{h(\tau_\sigma)} \in X$. Let (t_n) be the enumeration on $2^{<\mathbb{N}}$ according to the bijection φ and observe that $(x_{\sigma|n})$ is a subsequence of (x_{t_n}). By Lemma 28, we get that $(x_{\sigma|n})$ is a basic sequence which is weakly convergent to $x^{**}_{h(\tau_\sigma)} \in X$. This implies that $x^{**}_{h(\tau_\sigma)} = 0$, a contradiction. Therefore, $x^{**}_{h(\tau_\sigma)} \in X^{**} \setminus X$. With the same reasoning we verify that $y^{**}_{h(\tau_\sigma)} \in Y^{**} \setminus Y$.

Finally, let $\sigma, \sigma' \in 2^{\mathbb{N}}$ with $\sigma \neq \sigma'$. Notice that $\tau_\sigma \neq \tau_{\sigma'}$. The map h obtained by Lemma 23 is one-to-one. Therefore, $h(\tau_\sigma) \neq h(\tau_{\sigma'})$. By Lemma 22, we conclude that $x^{**}_{h(\tau_\sigma)} \neq x^{**}_{h(\tau_{\sigma'})}$ and $y^{**}_{h(\tau_\sigma)} \neq y^{**}_{h(\tau_{\sigma'})}$.

Having verified that the sequences $(x_t)_{t\in 2^{<\mathbb{N}}}$ and $(T(x_t))_{t\in 2^{<\mathbb{N}}}$ are both topologically equivalent to the basis of James tree, the proof of Theorem 3 is completed.

4. Proof of Theorem 4

This section is devoted to the proof of Theorem 4. Let us first argue that (ii) implies (i). If the operator T fixes a copy of ℓ_1, then clearly T^* has non-separable range. If, alternatively, there exists a bounded sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ in X such that its image $(T(x_t))_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree, then the non-separability of the range of T^* is guaranteed by Lemma 13.

We work now to prove that (i) implies (ii). As in the proof of Theorem 3, we fix a dense sequence (d_n) in B_X and we set $r_n = T(d_n)$ for every $n \in \mathbb{N}$. We distinguish the following cases.

CASE 1: *There exists a subsequence* (r_{l_n}) *of* (r_n) *which is equivalent to the standard unit vector basis of* ℓ_1. Let E be the closed subspace of X spanned by the corresponding subsequence (d_{l_n}) of (d_n). Notice that E is isomorphic to ℓ_1 and that $T : E \to Y$ is an isomorphic embedding. Hence, in this case we see that the operator T fixes a copy of ℓ_1.

CASE 2: *No subsequence of* (r_n) *is equivalent to the standard unit vector basis of* ℓ_1. We will show that there exists a bounded sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ in X such that its image $(T(x_t))_{t\in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree. The proof is similar to the proof of Theorem 3, and so, we shall only indicate the necessary changes.

First we observe that, by Rosenthal's Dichotomy [25] and our assumption, every subsequence of (r_n) has a further weak Cauchy subsequence. Therefore, by the Main Theorem in [26], the weak* closure $\mathcal{H}$ of the set $\{r_n\colon n \in \mathbb{N}\}$ in Y^{**} consists only of Baire-1 functions. We define the families $\mathcal{R}$ and $\mathcal{R}_0$ exactly as we did in (3) and (5) respectively. We recall that both $\mathcal{R}$ and $\mathcal{R}_0$ are hereditary and co-analytic. Moreover, $\mathcal{R}$ is cofinal in $[\mathbb{N}]^\infty$ while $\mathcal{R}_0$ is an M-family. Arguing as in the proof of Lemma 16, we see that there exists a hereditary, Borel and cofinal subfamily $\mathcal{F}'$ of $\mathcal{R}$. We define

$$\mathcal{A}' = \mathcal{F}' \setminus \mathcal{R}_0. \tag{14}$$

We have the following analogue of Lemma 18.

Lemma 29. *There exists a perfect Lusin gap inside* $(\mathcal{A}', \mathcal{R}_0)$.

Granting Lemma 29, the rest of the proof of Theorem 4 is the same to that of Theorem 3 mutatis mutandis.

So, what remains is to prove Lemma 29. By Theorem 8, it is enough to show that the family $\mathcal{A}'$ is not countably generated in the family $\mathcal{R}_0^\perp$. If this is not the case, then there exists a sequence (N_k) in $\mathcal{R}_0^\perp$ such that for every $L \in \mathcal{A}'$ there exists $k \in \mathbb{N}$ with $L \subseteq^* N_k$. For every $k \in \mathbb{N}$ let $\mathcal{H}_k$ be the weak* closure of the set $\{r_n\colon n \in N_k\}$ in Y^{**}. The fact that $N_k \in \mathcal{R}_0^\perp$ reduces to the fact that $0 \notin \mathcal{H}_k$. Therefore, there exist $E_k \subseteq Y^*$ finite and $\varepsilon_k > 0$ such that $\mathcal{H}_k \cap W(0, E_k, \varepsilon_k) = \emptyset$. Let E be the norm closure of the linear span of the set

$$E = \bigcup_k E_k.$$

The proof will be completed once we show that $T^*(E)$ is norm dense in $T^*(Y^*)$. To this end, we will argue by contradiction. So, assume that there exist $x^{**} \in X^{**}$, $y^* \in Y^*$ and $\delta > 0$ such that

(a) $\|x^{**}\| = \|T^*(y^*)\| = 1$,
(b) $T^*(E) \subseteq \mathrm{Ker}(x^{**})$, and
(c) $x^{**}(T^*(y^*)) > \delta$.

Since $T^{**}(B_{X^{**}}) \subseteq \mathcal{H}$ and $\mathcal{H}$ consists only of Baire-1 functions we may select $L \in \mathcal{R}$ such that the sequence $(r_n)_{n\in L}$ is weak* convergent to $T^{**}(x^{**})$. By (c) above, we may assume that $y^*(r_n) > \delta$ for every $n \in L$, and so, $[L]^\infty \cap \mathcal{R}_0 = \emptyset$. Since the family $\mathcal{F}'$ is cofinal in $[\mathbb{N}]^\infty$ and the sequence (N_k) generates $\mathcal{A}'$, it is possible to select $k_0 \in \mathbb{N}$, $y_0^* \in E_{k_0}$ and $A \in [L]^\infty$ such that $|y_0^*(r_n)| \geqslant \varepsilon_{k_0}$ for every $n \in A$. This implies that $T^*(y_0^*) \notin \mathrm{Ker}(x^{**})$ which contradicts property

(b) above. Having arrived to the desired contradiction the proof of Lemma 29 is completed, and as we have already indicated, the proof of Theorem 4 is also completed.

5. Comments

5.1. Theorem 3 and Theorem 4 were motivated by the structural results in [2,3] and our recent work on quotients of separable Banach spaces in [10] where a special case of Theorem 3 was proved and applied. Results of this type are, typically, used to reduce the existence of an *uncountable* family to the existence of a canonical *countable* object which is much more amenable to combinatorial manipulations.

5.2. We have already mentioned in the introduction that if an operator $T : X \to Y$ fixes a copy of a sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ topologically equivalent to the basis of James tree, then the topological structure of the weak* closure of $\{x_t\colon t \in 2^{<\mathbb{N}}\}$ in X^{**} is preserved under the action on T^{**}. Precisely, we have the following.

Lemma 30. *Let X and Y be Banach spaces and $T : X \to Y$ be an operator. Assume that there exists a sequence $(x_t)_{t\in 2^{<\mathbb{N}}}$ in X such that both $(x_t)_{t\in 2^{<\mathbb{N}}}$ and $(T(x_t))_{t\in 2^{<\mathbb{N}}}$ are topologically equivalent to the basis of James tree. Let $\mathcal{K}$ and $\mathcal{H}$ be the weak* closures of $\{x_t\colon t \in 2^{<\mathbb{N}}\}$ and $\{T(x_t)\colon t \in 2^{<\mathbb{N}}\}$ respectively. Then $T^{**} : \mathcal{K} \to \mathcal{H}$ is a weak* homeomorphism.*

Proof. Clearly we may assume that X and Y are separable. We observe that $\mathcal{K}$ and $\mathcal{H}$ consist only of Baire-1 functions.

Claim 31. *The weak* isolated points of $\mathcal{K}$ is the set $\{x_t\colon t \in 2^{<\mathbb{N}}\}$. Respectively, the weak* isolated points of $\mathcal{H}$ is the set $\{T(x_t)\colon t \in 2^{<\mathbb{N}}\}$.*

Proof of Claim 31. We will argue only for the set $\mathcal{K}$ (the argument for the set $\mathcal{H}$ is identical). Let $\mathcal{I}$ be the set of all weak* isolated points of $\mathcal{K}$. Let $x^{**} \notin \mathcal{I}$ be arbitrary and select an infinite subset A of $2^{<\mathbb{N}}$ such that the sequence $(x_t)_{t\in A}$ is weak* convergent to x^{**}. If A contains an infinite antichain, then $x^{**} = 0$. Otherwise, there exists $\sigma \in 2^{\mathbb{N}}$ such that $x^{**} = x^{**}_\sigma \in X^{**} \setminus X$. It follows that $\mathcal{I} \subseteq \{x_t\colon t \in 2^{<\mathbb{N}}\}$. To see the other inclusion assume, towards a contradiction, that there exists $s \in 2^{<\mathbb{N}}$ such that $x_s \notin \mathcal{I}$. Let (t_n) be the enumeration of $2^{<\mathbb{N}}$ according to the bijection φ. There exists a subsequence (x_k) of (x_{t_n}) which is weakly convergent to x_s. Since (x_k) is basic we get that $x_s = 0$, a contradiction. The proof of Claim 31 is completed. □

Claim 32. *For every infinite subset S of $2^{<\mathbb{N}}$ the sequence $(x_t)_{t\in S}$ is weak* convergent if and only if the sequence $(T(x_t))_{t\in S}$ is weak* convergent.*

Proof of Claim 32. Let E be a Banach space and $(e_t)_{t\in 2^{<\mathbb{N}}}$ be a sequence in E which is topologically equivalent to the basis of James tree. Let S be an arbitrary subset of $2^{<\mathbb{N}}$. By Definition 1, we see that the sequence $(e_t)_{t\in S}$ is weak* convergent if and only if either

(a) there exists an infinite antichain A of $2^{<\mathbb{N}}$ such that $S \subseteq^* A$ or
(b) there exists $\sigma \in 2^{\mathbb{N}}$ such that $S \subseteq^* \{\sigma \mid n\colon n \in \mathbb{N}\}$.

Using this observation, the result follows. □

By Claim 31, Claim 32 and the fact that $\mathcal{K}$ and $\mathcal{H}$ consist only of Baire-functions, we may apply Lemma 19 in [2] to infer that the map

$$\mathcal{K} \ni x_t \mapsto T(x_t) \in \mathcal{H}$$

is extended to a weak* homeomorphism $\Phi : \mathcal{K} \to \mathcal{H}$. Using the weak* continuity of T^{**} we see that $T^{**}|_{\mathcal{K}} = \Phi$. The proof of Lemma 30 is completed. □

5.3. Recall that a non-empty finite subset $\mathfrak{s}$ of $2^{<\mathbb{N}}$ is said to be a *segment* if there exist $s, t \in 2^{<\mathbb{N}}$ with $s \sqsubseteq t$ and such that $\mathfrak{s} = \{w \in 2^{<\mathbb{N}}: s \sqsubseteq w \sqsubseteq t\}$. Let $1 < p < +\infty$. The *p-James tree space*, denoted by JT_p, is defined to be the completion of $c_{00}(2^{<\mathbb{N}})$ under the norm

$$\|x\|_{JT_p} = \sup\left\{ \left(\sum_{i=0}^{d} \left| \sum_{t \in \mathfrak{s}_i} x(t) \right|^p \right)^{1/p} \right\}$$

where the above supremum is taken over all families $(\mathfrak{s}_i)_{i=0}^d$ of pairwise disjoint segments of $2^{<\mathbb{N}}$. The classical James tree space is the space JT_2.

Let $(e_t^p)_{t \in 2^{<\mathbb{N}}}$ be the standard Hamel basis of $c_{00}(2^{<\mathbb{N}})$ regarded as a sequence in JT_p. If (t_n) is the enumeration of $2^{<\mathbb{N}}$ according to the bijection φ, then the sequence $(e_{t_n}^p)$ defines a normalized Schauder basis of JT_p. It is easy to check that $(e_t^p)_{t \in 2^{<\mathbb{N}}}$ is topologically equivalent to the basis of James tree according to Definition 1 (a fact that actually justifies our terminology).

It is well known that the space JT_p is hereditarily ℓ_p; that is, every infinite-dimensional subspace of JT_p contains a copy of ℓ_p (see [17]). In particular, if $1 < p < q < +\infty$, then every operator $T : JT_p \to JT_q$ is strictly singular. Nevertheless, there do exist operators in $\mathcal{L}(JT_p, JT_q)$ fixing a copy of a sequence topologically equivalent to the basis of James tree. The natural inclusion map $I_{p,q} : JT_p \to JT_q$ is an example.

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